a0005 # Weakly Coupled Oscillators

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s0005 ## Introduction

p0005 Practically any physical, chemical, or biological system can exhibit rhythmic oscillatory activity, at least when the conditions are right. Winfree (2001) reviews the ubiquity of oscillations in nature, ranging from autocatalytic chemical reactions to pacemaker cells in the heart, to animal gates, and to circadian rhythms. When coupled, even weakly, oscillators interact via adjustment of their phases, that is, their timing, often leading to synchronization. In this chapter, we review the most important concepts needed to study and understand the dynamics of coupled oscillators.

p0010 From a mathematical point of view, an oscillator is a dynamical system,

$$\dot{x} = f(x), \quad x \in \mathbb{R}^m \qquad [1]$$

having a limit-cycle attractor – periodic orbit $\gamma \subset \mathbb{R}^m$. Its period is the minimal $T > 0$ such that

$$\gamma(t) = \gamma(t + T) \quad \text{for any } t$$

and its frequency is $\Omega = 2\pi/T$. Let $x(0) = x_0 \in \gamma$ be an arbitrary point on the attractor, then the state of the system, $x(t)$, is uniquely defined by its phase $\vartheta \in \mathbb{S}^1$ relative to x_0, where $\mathbb{S}^1$ is the unit circle.

p0015 Throughout this article, we assume that the periodic orbit γ is exponentially stable, which implies normal hyperbolicity. In this case, there is a continuous transformation $\Theta : U \to \mathbb{S}^1$ defined in a neighborhood $U \supset \gamma$ such that $\vartheta(t) = \Theta(x(t))$ for any trajectory in U, that is, Θ maps solutions of [1] to solutions of

$$\dot{\vartheta} = \Omega \qquad [2]$$

Such a transformation removes the amplitude but saves the phase of oscillation.

p0020 Accordingly, there is a continuous transformation that maps solutions of the weakly coupled network of n oscillators,

$$\dot{x}_i = f_i(x_i) + \varepsilon g_i(x_1, \ldots, x_n, \varepsilon), \quad \varepsilon \ll 1 \qquad [3]$$

onto solutions of the phase system

$$\dot{\vartheta}_i = \Omega_i + \varepsilon h_i(\vartheta_1, \ldots, \vartheta_n, \varepsilon), \quad \vartheta_i \in \mathbb{S}^1 \qquad [4]$$

which is easier for studying the collective properties of [3].

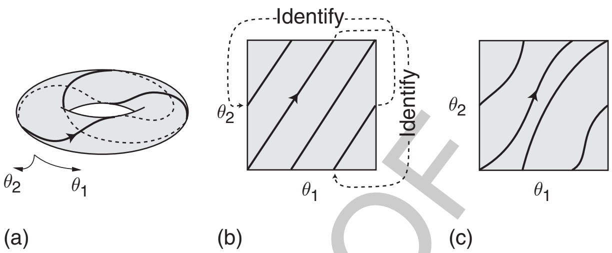


Figure 1 A 2-torus and its representation on the square. (Modified from Hoppensteadt and Izhikevich 1997.) f0005

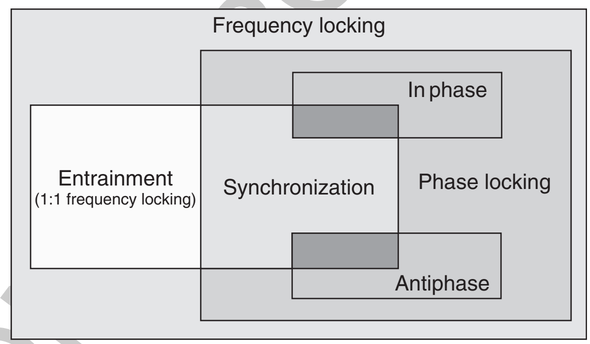
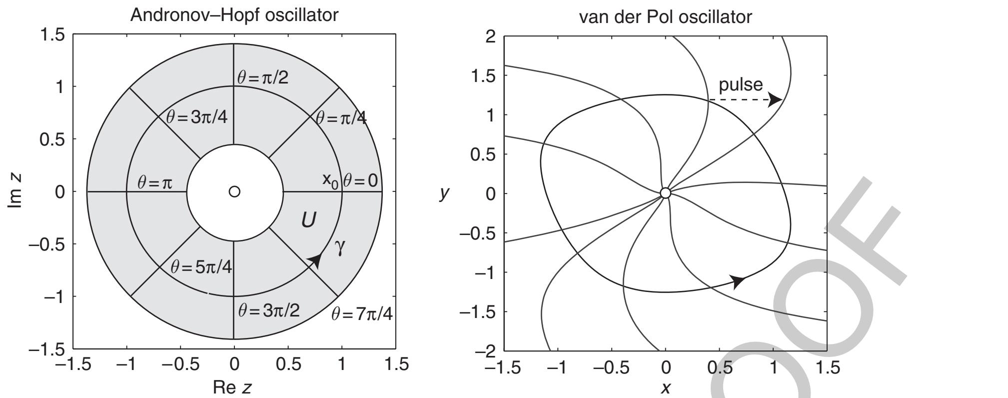


Figure 2 Various degrees of locking of oscillators. (Modified from Izhikevich 2006.) f0010

The oscillators are said to be frequency locked when [4] has a stable periodic orbit $\vartheta(t) = (\vartheta_1(t), \ldots, \vartheta_n(t))$ on the n-torus $\mathbb{T}^n$, as in **Figure 1a**. The "rotation vector" or "winding ratio" of the orbit is the set of integers $q_1 : q_2 : \cdots : q_n$ such that ϑ_1 makes q_1 rotations while ϑ_2 makes q_2 rotations, etc., as in the $2:3$ frequency locking in **Figure 1a**. The oscillators are entrained when they are $1:1:\cdots:1$ frequency locked. The oscillators are phase locked when there is an $(n-1) \times n$ integer matrix K having linearly independent rows such that $K\vartheta(t) = \text{const}$. For example, the two oscillators in **Figure 1b** are phase locked with $K = (2, 3)$, while those in **Figure 1c** are not. The oscillators are synchronized when they are entrained and phase locked. Synchronization is in-phase when $\vartheta_1(t) = \cdots = \vartheta_n(t)$ and out-of-phase otherwise. Two oscillators are said to be synchronized antiphase when $\vartheta_1(t) - \vartheta_2(t) = \pi$. Frequency locking without phase locking, as in **Figure 1c**, is called phase trapping. The relationship between all these definitions is depicted in **Figure 2**. p0025

Phase Resetting s0010

An exponentially stable periodic orbit is a normally hyperbolic invariant manifold, hence its sufficiently small neighborhood, U, is invariantly foliated by p0030



f0015 **Figure 3** Isochrons of Andronov–Hopf oscillator ($\dot{z}=(1+i)z-z|z|^2$, $z\in C$) and van der Pol oscillator ($\dot{x}=x-x^3-y$, $\dot{y}=x$).

stable submanifolds (Guckenheimer 1975) illustrated in **Figure 3**. The manifolds represent points having equal phases and, for this reason, they are called isochrons (from Greek "iso" meaning equal and "chronos" meaning time).

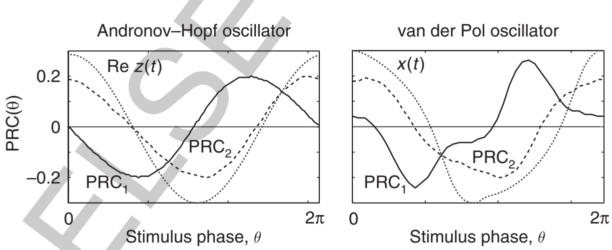
p0035 The geometry of isochrons determines how the oscillators react to perturbations. For example, the pulse in **Figure 3**, right, moves the trajectory from one isochron to another, thereby changing its phase. The magnitude of the phase shift depends on the amplitude and the exact timing of the stimulus relative to the phase of oscillation ϑ. Stimulating the oscillator at different phases, one can measure the phase transition curve (Winfree 2001)

$$\vartheta_{\text{new}} = \text{PTC}(\vartheta_{\text{old}})$$

and the phase resetting curve

$$\text{PRC}(\vartheta) = \text{PTC}(\vartheta) - \vartheta$$
$$(\text{shift} = \text{new phase} - \text{old phase})$$

Positive (negative) values of the PRC correspond to phase advances (delays). PRCs are convenient when the phase shifts are small, so that they can be magnified and clearly seen, as in **Figure 4**. PTCs are convenient when the phase shifts are large and comparable with the period of oscillation.



f0020 **Figure 4** Examples of phase response curves (PRCs) of the oscillators in **Figure 3**. $\text{PRC}_1(\vartheta)$ and $\text{PRC}_2(\vartheta)$ correspond to horizontal (along the first variable) and vertical (along the second variable) pulses with amplitudes 0.2. An example of oscillation is plotted as a dotted curve in each subplot (not to scale).

In **Figure 5** we depict phase portraits of the Andronov–Hopf oscillator receiving pulses of magnitude 0.5 (left) and 1.5 (right). Notice the drastic difference between the corresponding PRCs or PTCs. Winfree (2001) distinguishes two cases: p0040

1. type 1 (weak) resetting results in continuous PRCs and PTCs with mean slope 1, and
2. type 0 (strong) resetting results in discontinuous PRCs and PTCs with mean slope 0.

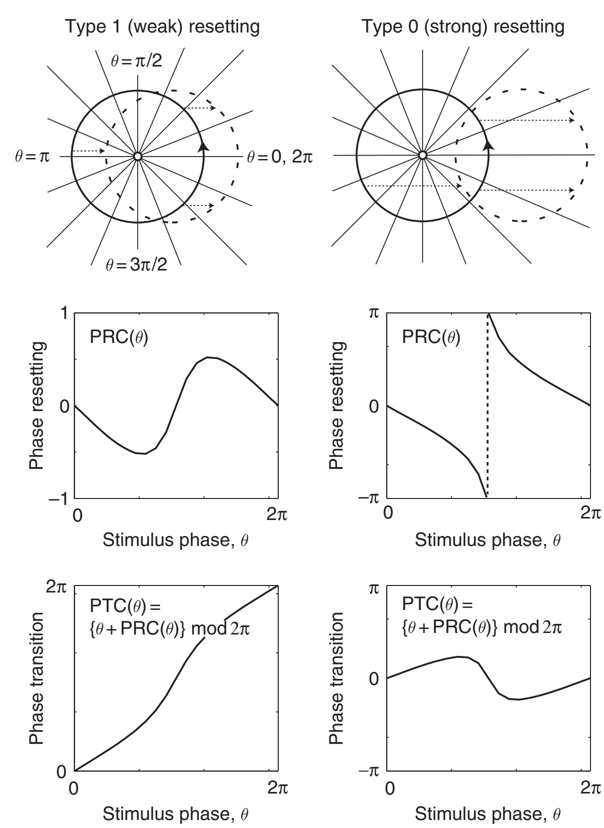


Figure 5 Types of phase resetting of the Andronov–Hopf oscillator in **Figure 3**. f0025

The discontinuity of type 0 PRC in **Figure 5** is a topological property that cannot be removed by reallocating the initial point x_0 that corresponds to zero phase. The discontinuity stems from the fact that the shifted image of the limit cycle (dashed circle) goes beyond the central equilibrium at which the phase is not defined.

p0045 The stroboscopic mapping of $\mathbb{S}^1$ to itself, called Poincaré phase map,

$$\vartheta_{k+1} = \mathrm{PTC}(\vartheta_k) \qquad [5]$$

describes the response of an oscillator to a T-periodic pulse train. Here, ϑ_k denotes the phase of oscillation when the kth input pulse arrives. Its fixed points correspond to synchronized solutions, and its periodic orbits correspond to phase-locked states.

s0015 Weak Coupling

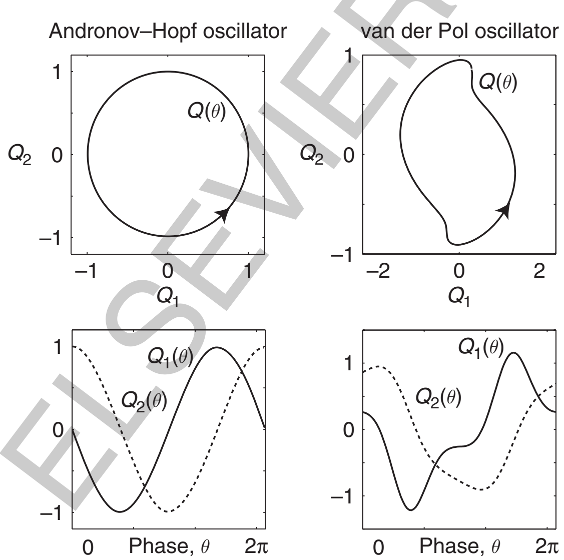
p0050 Now consider dynamical systems of the form

$$\dot{x} = f(x) + \varepsilon s(t) \qquad [6]$$

describing periodic oscillators, $\dot{x} = f(x)$, forced by a weak time-depended input $\varepsilon s(t)$, for example, from other oscillators in a network. Let $\Theta(x)$ denote the phase of oscillation at point $x \in U$, so that the map $\Theta : U \to \mathbb{S}^1$ is constant along each isochron. This mapping transforms [6] into the phase model

$$\dot{\vartheta} = \Omega + \varepsilon Q(\vartheta) \cdot s(t)$$

with function $Q(\vartheta)$, illustrated in **Figure 6**, satisfying three equivalent conditions:



f0030 **Figure 6** Solutions $Q = (Q_1, Q_2)$ to the adjoint problem [7] for oscillators in **Figure 3**.

1. Winfree: $Q(\vartheta)$ is normalized PRC to infinitesimal pulsed perturbations;
2. Kuramoto: $Q(\vartheta) = \mathrm{grad}\ \Theta(x)$; and
3. Malkin: Q is the solution to the adjoint problem

$$\dot{Q} = -\{Df(\gamma(t))\}^{\top} Q \qquad [7]$$

with the normalization $Q(t) \cdot f(\gamma(t)) = \Omega$ for any t.

The function $Q(\vartheta)$ can be found analytically in a few simple cases: p0055

1. a nonlinear phase oscillator $\dot{x} = f(x)$ with $x \in \mathbb{S}^1$ and $f > 0$ has $Q(\vartheta) = \Omega / f(\gamma(\vartheta))$;
2. a system near saddle-node on invariant circle bifurcation has $Q(\vartheta)$ proportional to $1 - \cos\vartheta$; and
3. a system near supercritical Andronov–Hopf bifurcation has $Q(\vartheta)$ proportional to $\sin(\vartheta - \psi)$, where $\psi \in \mathbb{S}^1$ is a constant phase shift.

Other interesting cases, including homoclinic, relaxation, and bursting oscillators are considered by Izhikevich (2006).

Treating $s(t)$ in [6] as the input from the network, we can transform weakly coupled oscillators p0060

$$\dot{x}_i = f_i(x_i) + \varepsilon \overbrace{\sum_{j=1}^{n} g_{ij}(x_i, x_j)}^{s_i(t)}, \quad x_i \in \mathbb{R}^m \qquad [8]$$

to the phase model

$$\dot{\vartheta}_i = \Omega_i + \varepsilon\, Q_i(\vartheta_i) \cdot \overbrace{\sum_{j=1}^{n} g_{ij}(x_i(\vartheta_i), x_j(\vartheta_j))}^{s_i(t)} \qquad [9]$$

having the form [4] with $h_i = Q_i \sum g_{ij}$, or the form

$$\dot{\vartheta}_i = \Omega_i + \varepsilon \sum_{j=1}^{n} h_{ij}(\vartheta_i, \vartheta_j)$$

where $h_{ij} = Q_i g_{ij}$. Introducing phase deviation variables $\vartheta_i = \Omega_i t + \varphi_i$, we transform this system into the form

$$\dot{\varphi}_i = \varepsilon \sum_{j=1}^{n} h_{ij}(\Omega_i t + \varphi_i,\ \Omega_j t + \varphi_j)$$

which can be averaged to

$$\dot{\varphi}_i = \varepsilon \sum_{j=1}^{n} H_{ij}(\varphi_i - \varphi_j) \qquad [10]$$

with the functions

$$H_{ij}(\chi) = \lim_{T\to\infty} \frac{1}{T} \int_0^T h_{ij}(\Omega_i t,\ \Omega_j t - \chi)\, \mathrm{d}t \qquad [11]$$

describing the interaction between oscillators (Ermentrout and Kopell 1984). To summarize, we transformed weakly coupled system [8] into the phase model [10] with H given by [11] and each Q being the solution to the adjoint problem [7]. This constitutes the Malkin theorem for weakly coupled oscillators (Hoppensteadt and Izhikevich 1997, theorem 9.2).

p0065 Existence of one equilibrium of the phase model [10] implies the existence of the entire circular family of equilibria, since translation of all φ_i by a constant phase shift does not change the phase differences $\varphi_i - \varphi_j$ and hence the form of [10]. This family corresponds to a limit cycle of [8], on which all oscillators have equal frequencies and constant phase shifts, that is, they are synchronized, possibly out of phase.

p0070 We say that two oscillators, i and j, have resonant (or commensurable) frequencies when the ratio Ω_i/Ω_j is a rational number, for example, it is p/q for some integer p and q. They are nonresonant when the ratio is an irrational number. In this case, the function H_{ij} defined above is constant regardless of the details of the oscillatory dynamics or the details of the coupling, that is, dynamics of two coupled nonresonant oscillators is described by an uncoupled phase model. Apparently, such oscillators do not interact; that is, the phase of one of them cannot change the phase of the other one even on the long timescale of order $1/\varepsilon$.

s0020 Synchronization

p0075 Consider [8] with $n = 2$, describing two mutually coupled oscillators. Let us introduce "slow" time $\tau = \varepsilon t$ and rewrite the corresponding phase model [10] in the form

$$\varphi_1' = \omega_1 + H_{12}(\varphi_1 - \varphi_2)$$
$$\varphi_2' = \omega_2 + H_{21}(\varphi_2 - \varphi_1)$$

where $' = \mathrm{d}/\mathrm{d}\tau$ and $\omega_i = H_{ii}(0)$ is the frequency deviation from the natural oscillation, $i = 1, 2$. Let $\chi = \varphi_2 - \varphi_1$ denote the phase difference between the oscillators; then

$$\chi' = \omega + H(\chi) \qquad [12]$$

where

$$\omega = \omega_2 - \omega_1 \text{ and } H(\chi) = H_{21}(\chi) - H_{12}(-\chi)$$

is the frequency mismatch and the antisymmetric part of the coupling, respectively, illustrated in **Figure 7**, dashed curves. A stable equilibrium of [12] corresponds to a stable limit cycle of the phase model.

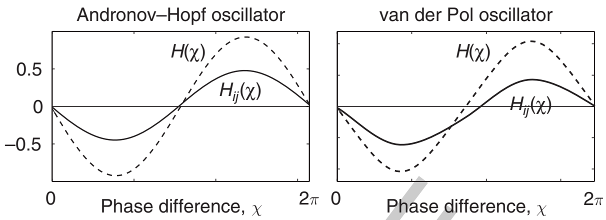


Figure 7 Solid curves: functions $H_{ij}(\chi)$ defined by [11] corresponding to the gap–junction input $g(x_i, x_j) = (x_{j1} - x_{i1}, 0)$. Dashed curves: functions $H(\chi) = H_{ji}(\chi) - H_{ij}(-\chi)$.. Parameters are as in **Figure 3**. f0035

All equilibria of [12] are solutions to $H(\chi) = -\omega$, and they are intersections of the horizontal line $-\omega$ with the graph of H. They are stable if the slope of the graph is negative at the intersection. If oscillators are identical, then $H(\chi)$ is an odd function (i.e., $H(-\chi) = -H(\chi)$), and $\chi = 0$ and $\chi = \pi$ are always equilibria, possibly unstable, corresponding to the in-phase and antiphase synchronized solutions. The in-phase synchronization of gap–junction coupled oscillators in **Figure 7** is stable because the slope of H (dashed curves) is negative at $\chi = 0$. The max and min values of the function H determine the tolerance of the network to the frequency mismatch ω, since there are no equilibria outside this range. p0080

Now consider a network of $n > 2$ weakly coupled oscillators [8]. To determine the existence and stability of synchronized states in the network, we need to study equilibria of the corresponding phase model [10]. The vector $\phi = (\phi_1, \ldots, \phi_n)$ is an equilibrium of [10] when p0085

$$0 = \omega_i + \sum_{j \neq 1}^{n} H_{ij}(\phi_i - \phi_j) \quad (\text{for all } i) \qquad [13]$$

It is stable when all eigenvalues of the linearization matrix (Jacobian) at ϕ have negative real parts, except one zero eigenvalue corresponding to the eigenvector along the circular family of equilibria (ϕ plus a phase shift is a solution of [13] too since the phase shifts $\phi_j - \phi_i$ are not affected).

In general, determining the stability of equilibria is a difficult problem. Ermentrout (1992) found a simple sufficient condition. If p0090

1. $a_{ij} = H'_{ij}(\phi_i - \phi_j) \leq 0$, and
2. the directed graph defined by the matrix $a = (a_{ij})$ is connected, (i.e., each oscillator is influenced, possibly indirectly, by every other oscillator),

then the equilibrium ϕ is neutrally stable, and the corresponding limit cycle $x(t + \phi)$ of [8] is asymptotically stable.

p0095 Another sufficient condition was found by Hoppensteadt and Izhikevich (1997). If system [10] satisfies

1. $\omega_1 = \cdots = \omega_n = \omega$ (identical frequencies)
2. $H_{ij}(-\chi) = -H_{ji}(\chi)$ (pairwise odd coupling)

for all i and j, then the network dynamics converge to a limit cycle. On the cycle, all oscillators have equal frequencies $1 + \varepsilon\omega$ and constant phase deviations.

p0100 The proof follows from the observation that [10] is a gradient system in the rotating coordinates $\varphi = \omega\tau + \phi$ with the energy function

$$E(\phi) = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} R_{ij}(\phi_i - \phi_j)$$

where

$$R_{ij}(\chi) = \int_0^{\chi} H_{ij}(s)\,\mathrm{d}s$$

One can check that $\mathrm{d}E(\phi)/\mathrm{d}\tau = -\sum(\phi_i')^2 \leq 0$ along the trajectories of [12] with equality only at equilibria.

s0025 Mean-Field Approximations

p0105 Let us represent the phase model [10] in the form

$$\varphi_i' = \omega_i + \sum_{j\neq i}^{n} H_{ij}(\varphi_i - \varphi_j)$$

where $' = \mathrm{d}/\mathrm{d}\tau, \tau = \varepsilon t$ is the slow time, and $\omega_i = H_{ii}(0)$ are random frequency deviations. Collective dynamics of this system can be analyzed in the limit $n \to \infty$. We illustrate the theory using the special case, $H(\chi) = -\sin\chi$, known as the Kuramoto (1984) model:

$$\varphi_i' = \omega_i + \frac{K}{n}\sum_{j=1}^{n}\sin(\varphi_j - \varphi_i), \quad \varphi_i \in [0, 2\pi] \qquad [14]$$

where $K > 0$ is the coupling strength and the factor $1/n$ ensures that the model behaves well as $n \to \infty$. The complex-valued sum of all phases,

$$re^{i\psi} = \frac{1}{n}\sum_{j=1}^{n} e^{i\varphi_j}$$

(Kuramoto synchronization index) [15]

describes the degree of synchronization in the network. Apparently, the in-phase synchronized state $\varphi_1 = \cdots = \varphi_n$ corresponds to $r = 1$ with ψ being the population phase. In contrast, the incoherent state with all φ_i having different values randomly distributed on the unit circle, corresponds to $r \approx 0$. Intermediate values of r correspond to a partially synchronized or coherent state, depicted in **Figure 8**. Some phases are synchronized forming a cluster, while others roam around the circle.

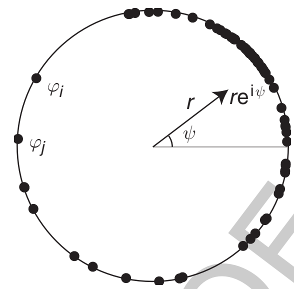


Figure 8 Kuramoto synchronization index [15] describes the degree of coherence in the network [14]. f0040

Multiplying both sides of [15] by $e^{-i\varphi_i}$ and considering only the imaginary parts, we can rewrite [14] in the equivalent form p0110

$$\varphi_i' = \omega_i + Kr\sin(\psi - \varphi_i)$$

that emphasizes the mean-filed character of interactions between the oscillators: they all are pulled into the synchronized cluster ($\varphi_i \to \psi$) with the effective strength proportional to the cluster size r. This pull is offset by the random frequency deviations ω_i that pull away from the cluster.

Let us assume that ω_i's are distributed randomly around 0 with a symmetrical probability density function $g(\omega)$, for example, Gaussian. Kuramoto has shown that in the limit $n \to \infty$, the cluster size r obeys the self-consistency equation p0115

$$r = rK\int_{-\pi/2}^{+\pi/2} g(Kr\sin\varphi)\cos^2\varphi\,\mathrm{d}\varphi \qquad [16]$$

Notice that $r = 0$, corresponding to the incoherent state, is always a solution of this equation. When the coupling strength K is greater than a certain critical value,

$$K_c = \frac{2}{\pi g(0)}$$

an additional, nontrivial solution $r > 0$ appears, which corresponds to a partially synchronized state. Expanding g in a Taylor series, one gets the scaling $r = \sqrt{16(K - K_c)/(-g''(0)\pi K_c^4)}$. Thus, the stronger the coupling K relative to the random distribution of frequencies, the more oscillators synchronize into a coherent cluster. The issue of stability of incoherent and partially synchronized states is discussed by Strogatz (2000). Other generalizations of the Kuramoto model are reviewed by Acebron *et al.* (2005). An extended version of this article with the emphasis on computational neuroscience can be found in the recent book by Izhikevich (2006).

Further Reading

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Wheeler–De Witt Theory

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Introduction

p0005 It is recognized that one of the outstanding problems in modern physics is to formulate the quantum theory of gravity, synthesizing the principles of quantum mechanics and general theory of relativity. The fundamental units for measuring time, length, and energy, known as Planck time, Planck length, and Planck energy, respectively, are defined to be $t_{\mathrm{Pl}} = (\hbar G/c^5)^{1/2} = 5.39\times10^{-44}\,\mathrm{s}$, $l_{\mathrm{Pl}} = (\hbar G/c^3)^{1/2} = 1.61\times10^{-33}\,\mathrm{cm}$, and $m_{\mathrm{Pl}} = (\hbar c/G)^{1/2} = 2.17\times10^{-5}\,\mathrm{g}$, in terms of the Newton's constant, G, velocity of light, c, and $\hbar = h/2\pi, h$ being the Planck's constant. We may conclude, on dimensional arguments, that quantum gravity effects will play an important role when we consider physical phenomena in the vicinity of these scales. Therefore, when we probe very short distances, consider collisions at Planckian energies, and envisage evolution of the universe in the Planck era, the quantum gravity will come into play in a predominant manner. The purpose of this article is to present an overview of an approach to quantize Einstein's theory of gravity, pioneered by Wheeler and De Witt almost four decades ago. We proceed to recapitulate various prescriptions for quantizing gravitation and then discuss simple derivation of the Wheeler–De Witt (WDW) equation in general relativity and some of its applications in the study of quantum cosmology. There are, broadly speaking, three different approaches to quantize gravity.

The general theory of relativity has been tested to great degree of accuracy in the classical regime. The geometrical description of spacetime plays a cardinal role in Einstein's theory. Therefore, the general relativists emphasize the geometrical attributes of the theory and the central role played by the spacetime structure in their formulation of quantum theory. It is natural to adopt a background-independent approach. In contrast, the path followed by quantum field theorists, where the prescription is valid in the weak-field approximation, the theory is quantized in a given background, usually the Minkowskian space. It is argued by the proponents of the geometric approach, that the background metric should emerge from the theory in a self-consistent manner rather than being introduced by hand when we quantize the theory. One of the earliest attempts to quantize gravity was to follow the route of canonical method. The canonical quantization approach has many advantages. One of the important features is that it is quite similar to the prescriptions adopted in quantum field theory where one uses notion of operators, commutation relations, etc. Moreover, the subtleties encountered in quantizing gravity are transparent. Therefore, the canonical procedure is preferred over the path-integral formulation, although the latter has its own advantages too. Another positive aspect of the canonical approach is that the requirement of background-independent formulation could be maintained to some extent. Thus, there is room for p0010